Section 001 CDS 101 *Introduction to Computational and Data Sciences*

Class 21: Statistical distributions II

April 10, 2018

General

Annoucements

- Reading for next class: *Introductory Statistics with Randomization and Simulation*
    - From chapter 2: from the beginning through to the end of section 2.2
- Homework 3 posted, due next Monday, April 16th by 11:59pm.

Statistical computations in R

Useful statistical functions

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- `median()`: Computes the median
- `min()`: Finds the minimum value
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- `sd()`: Computes the standard deviation
- `IQR()`: Computes the interquartile range
- `quantile()`: Computes quantiles (percentiles)

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county %>%
  summarize(
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    min = min(mean_work_travel),
    max = max(mean_work_travel),
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```

mean	median	min	max	sd	iqr
22.72558	22.4	4.3	44.2	5.514159	7.1

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```
county %>%
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    Q2 = quantile(mean_work_travel, probs = c(0.50), type = 1),
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```

Q1	Q2	Q3	Q4
19	22.4	26.1	44.2

From histograms to probability mass functions

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mean_work_travel
25.1
25.8
23.8
28.3
33.2
28.1
25.1
...

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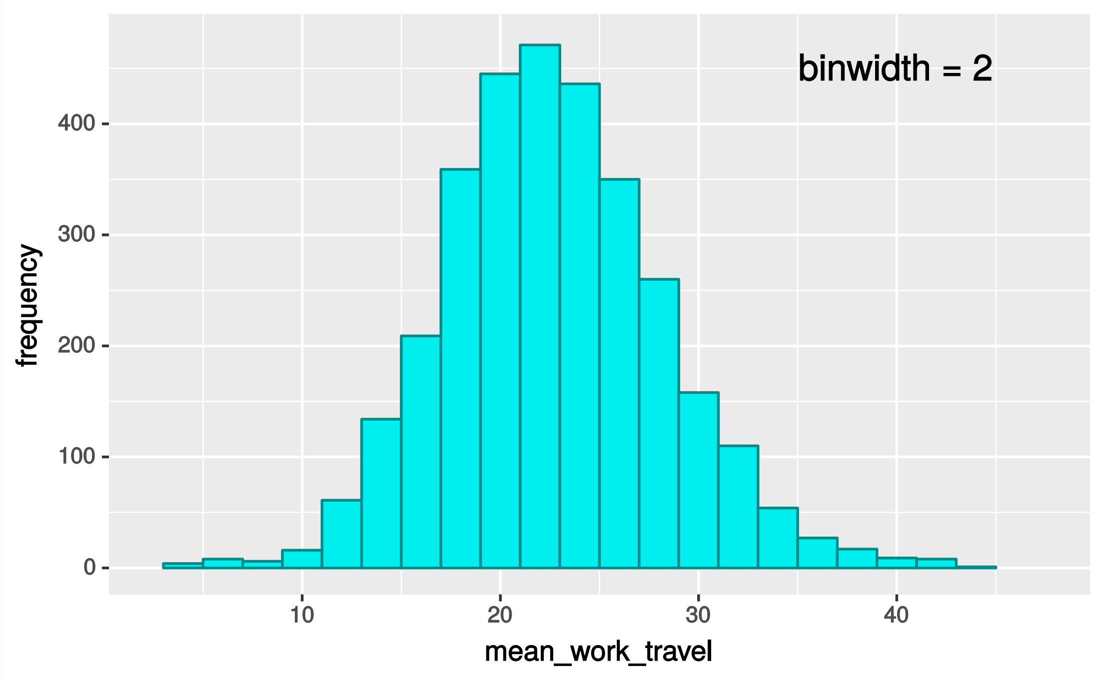

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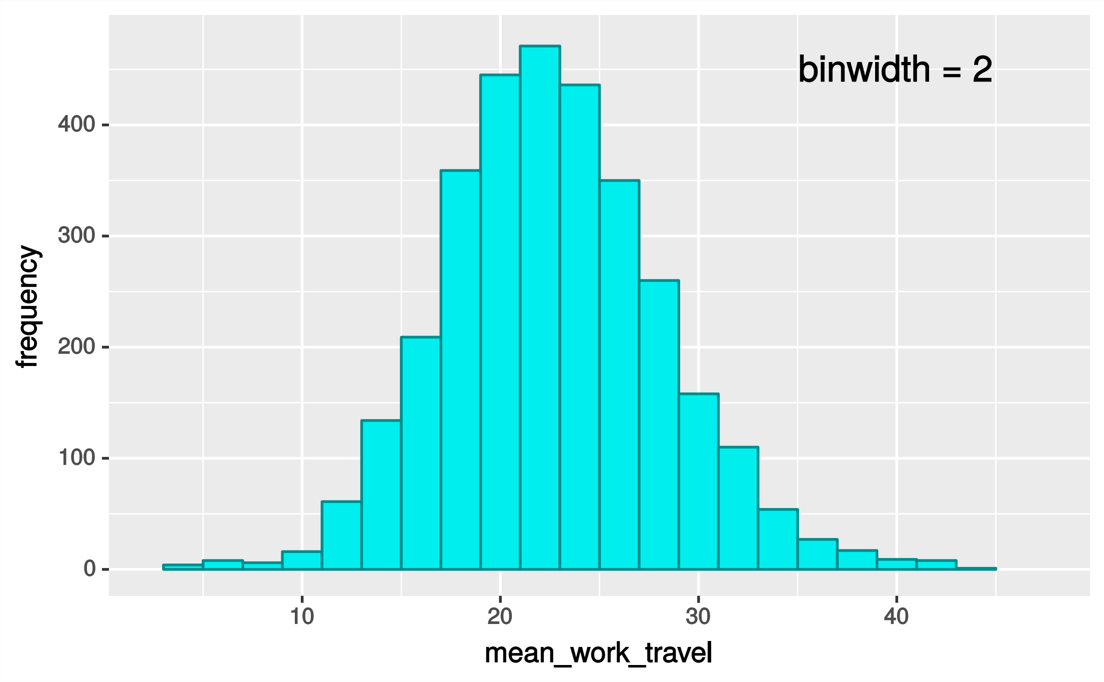

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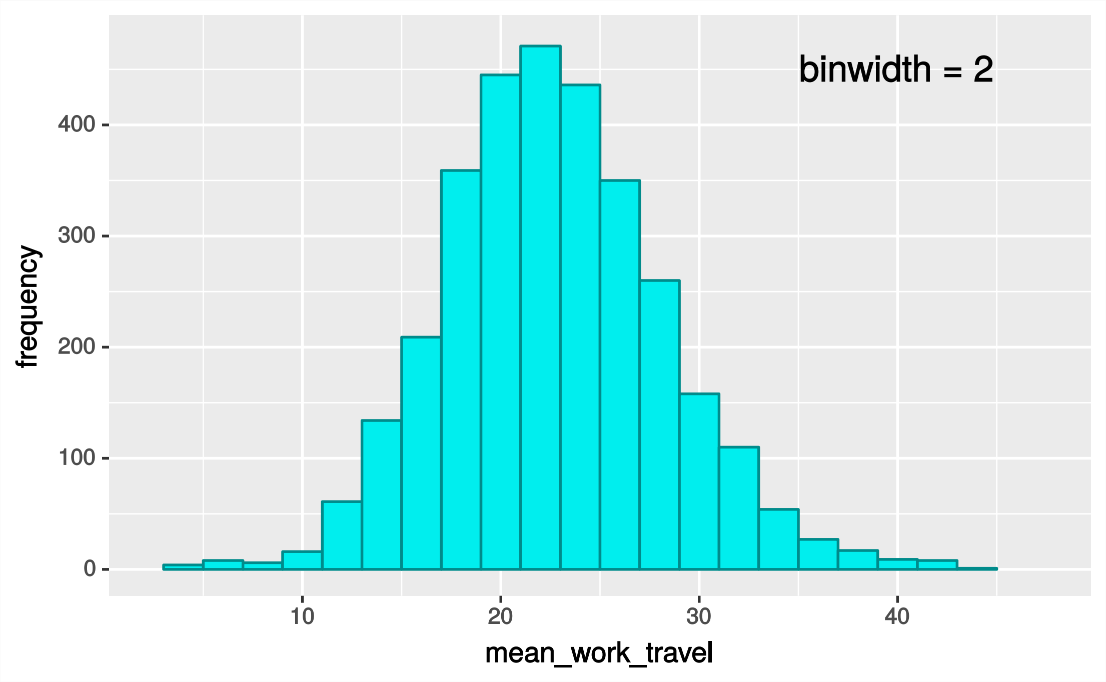

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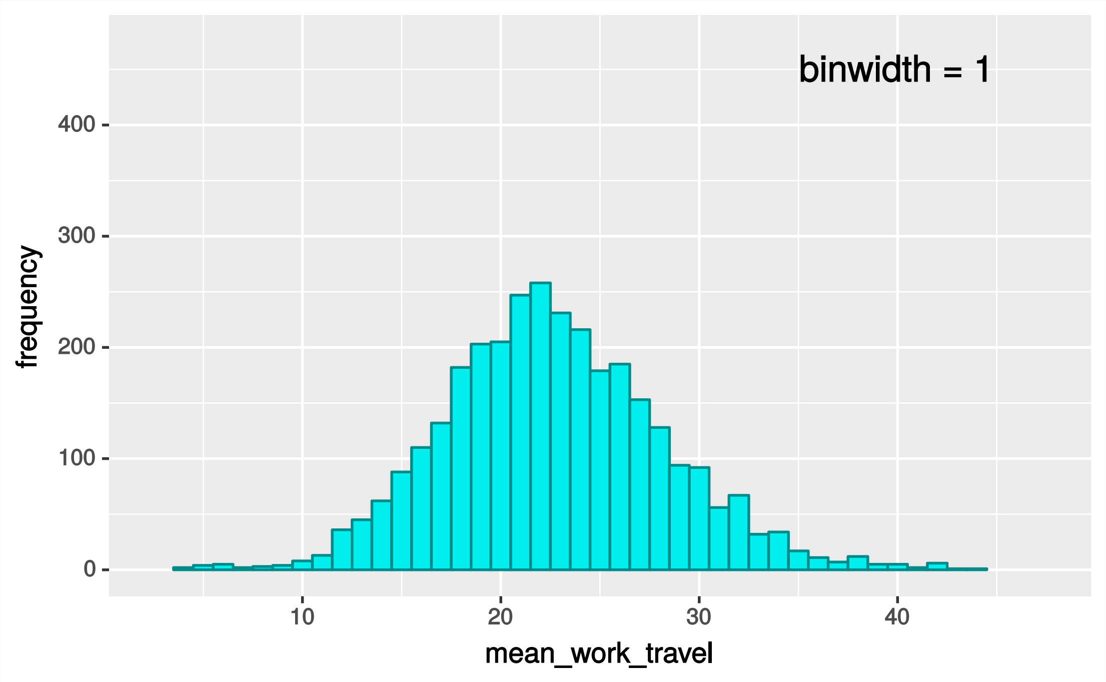

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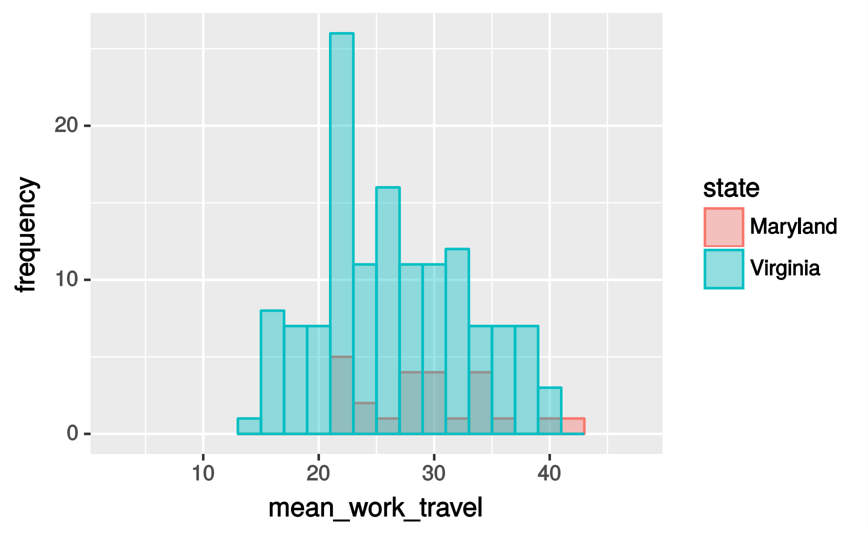

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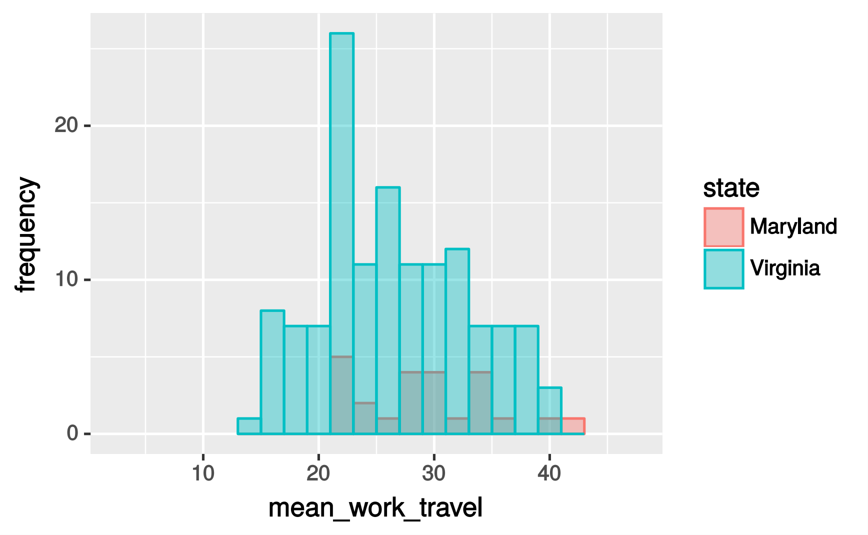

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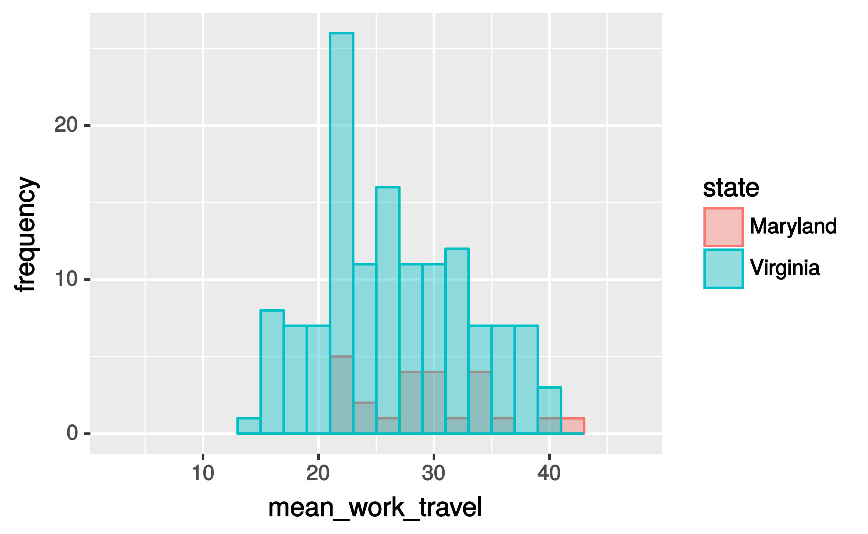

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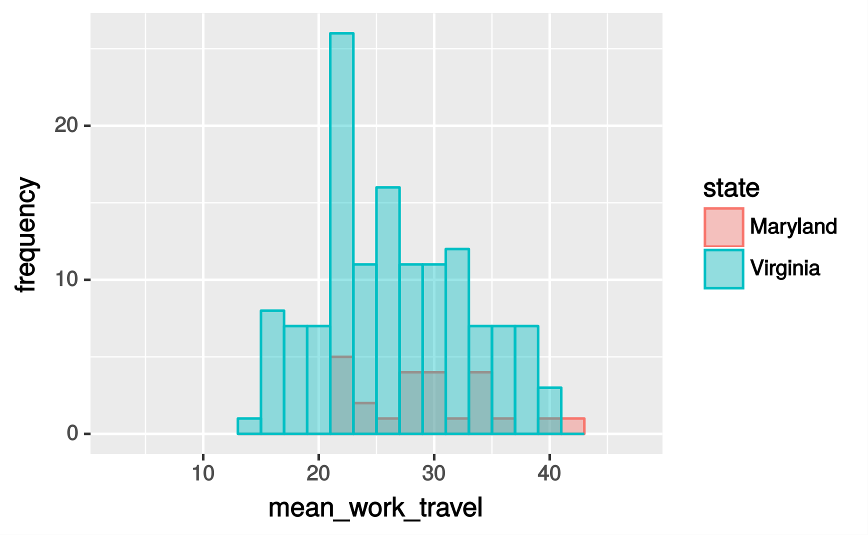

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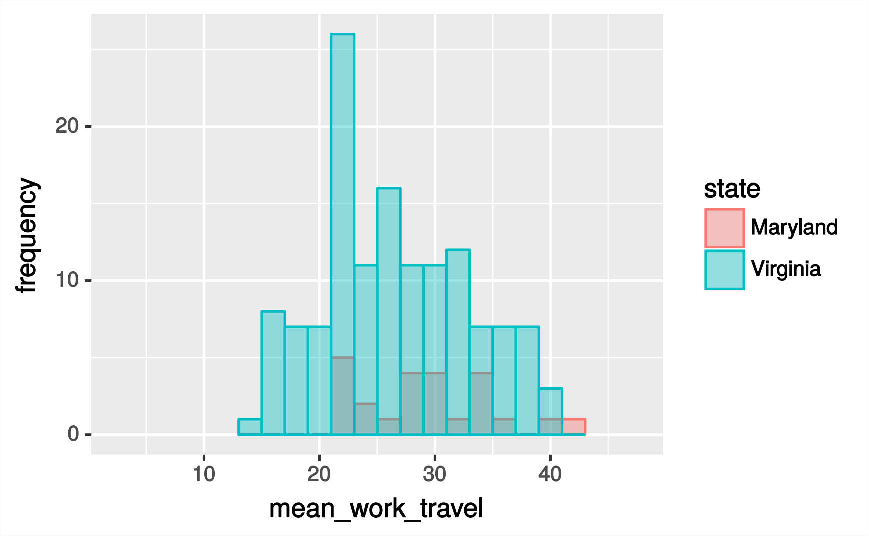

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- We need to **normalize** the frequency counts

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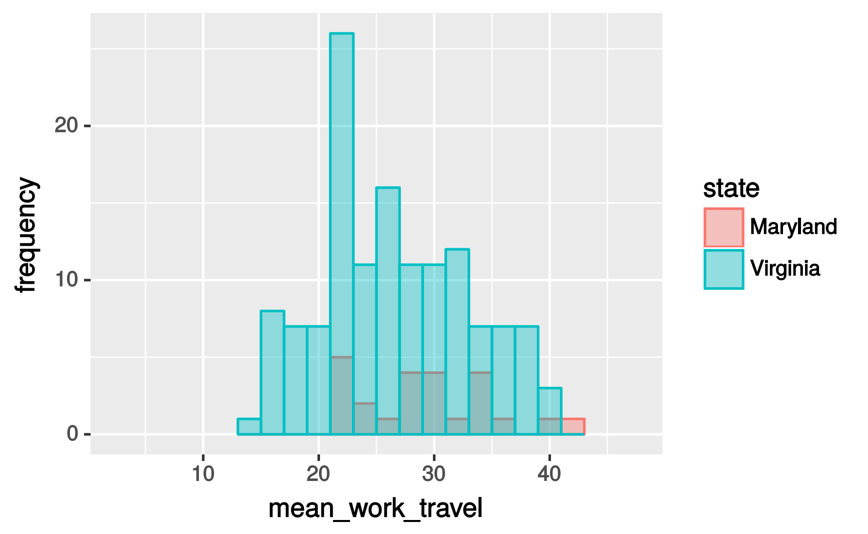

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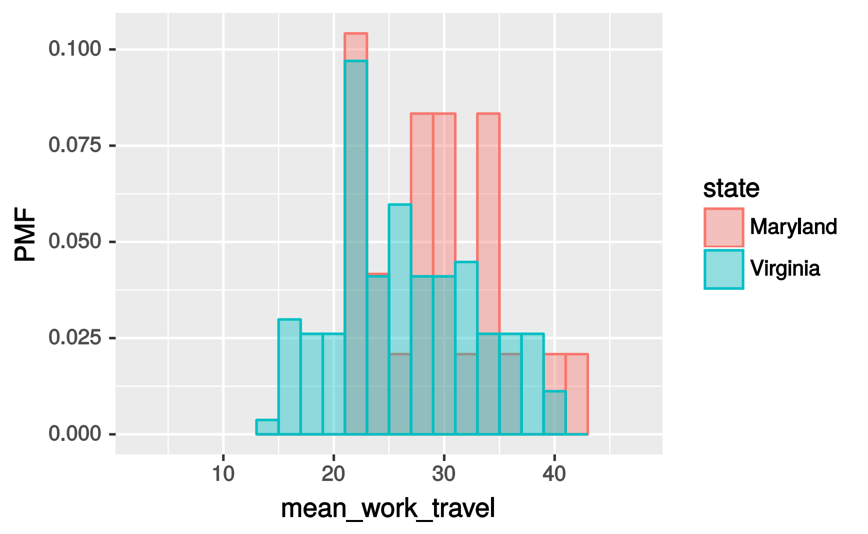
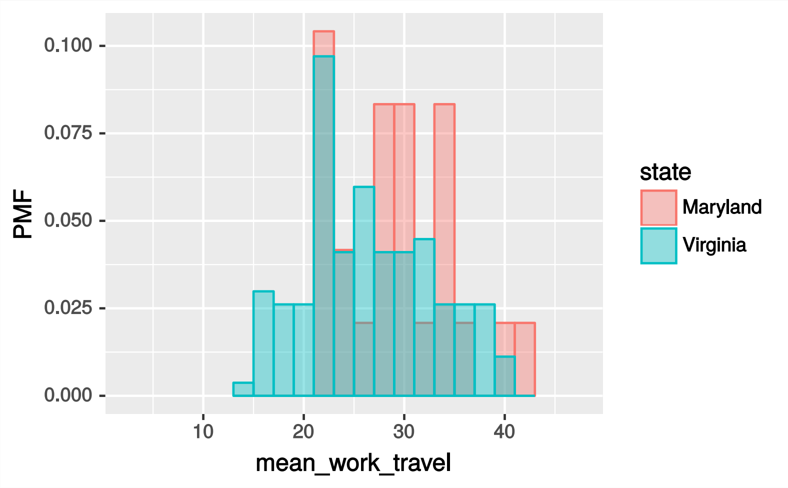


Probability mass function (PMF)



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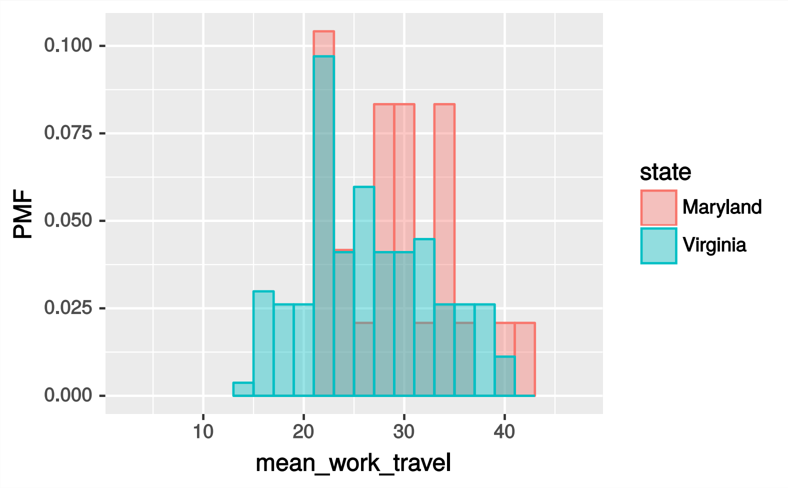

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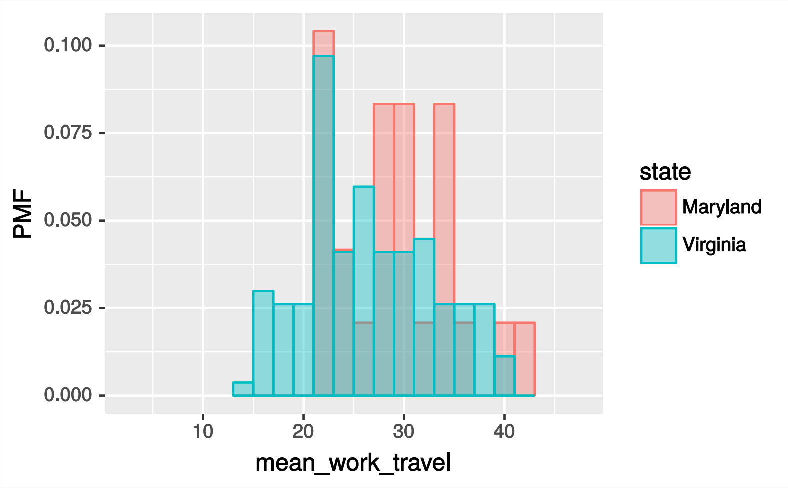

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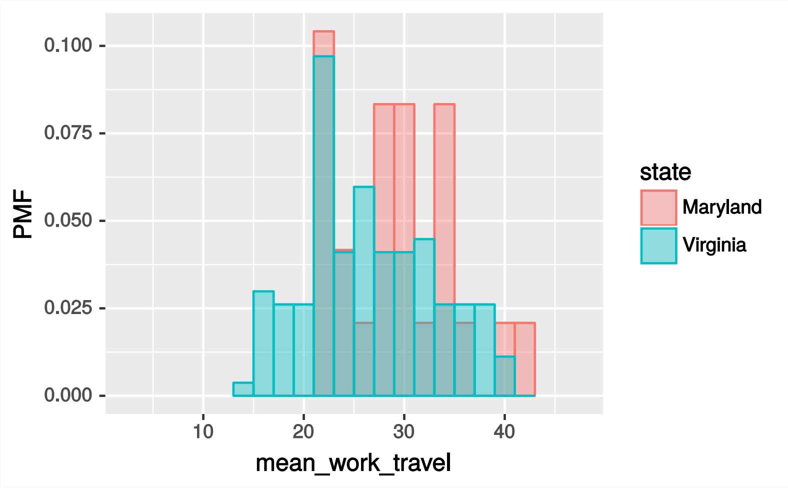

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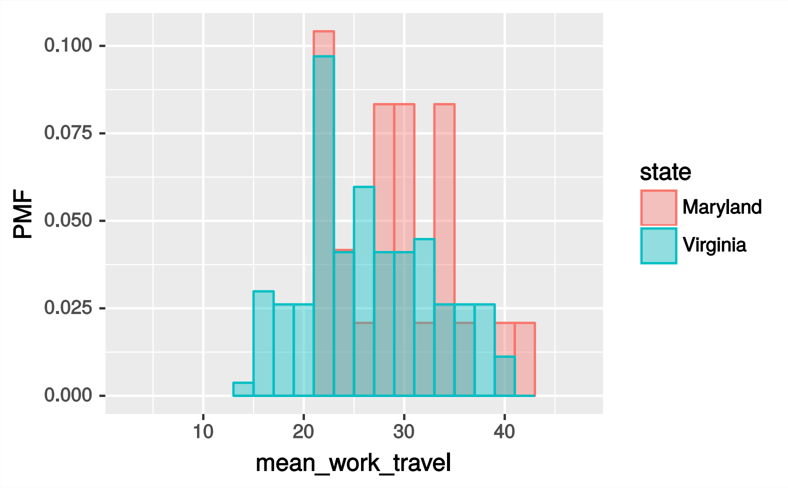

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  filter(state == "Virginia" | state == "Maryland") %>%
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  geom_histogram(
    mapping = aes(x = mean_work_travel, fill = state),
    binwidth = 2, center = 0, position = "identity", alpha = 0.4) +
  labs(y = "frequency") +
  coord_cartesian(xlim = c(2.5, 47.5))
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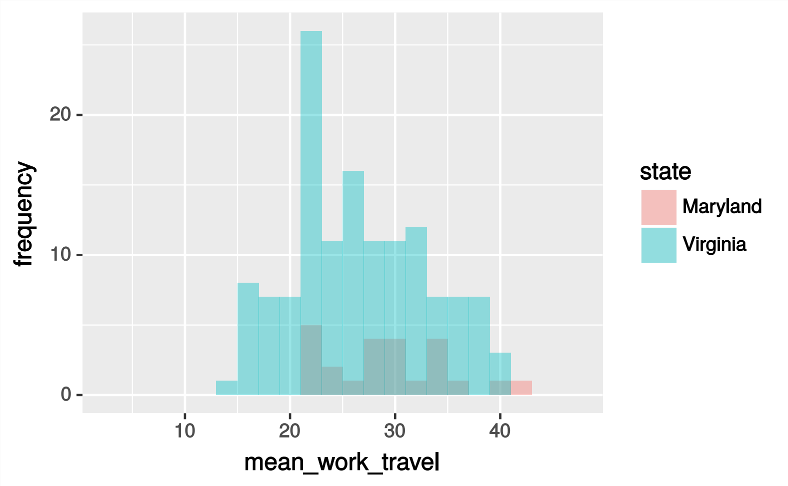

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county %>%
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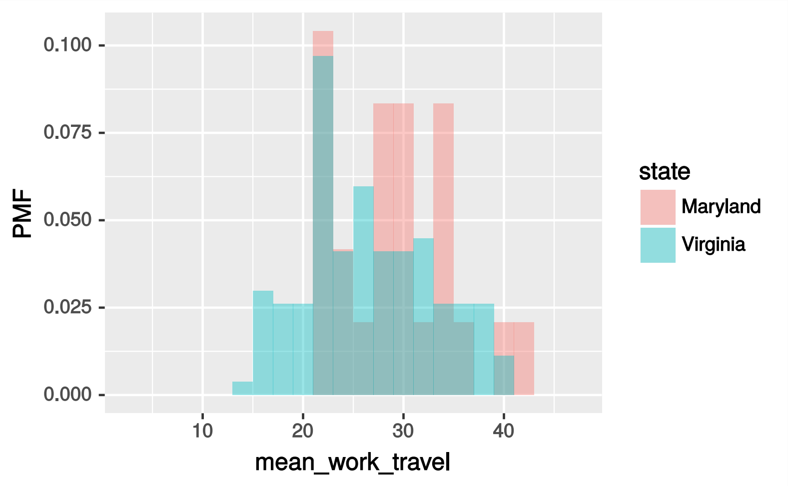

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Assign the figure to a variable

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va_md_pmf_figure <- county %>%
  filter(state == "Virginia" | state == "Maryland") %>%
  ggplot() +
  geom_histogram(
    mapping = aes(x = mean_work_travel, y = ..density.., fill = state),
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```

Use `ggplot_build()` with `pluck()` and `as_tibble()` as follows:

```
va_md_pmf_data <- va_md_pmf_figure %>%
  ggplot_build() %>%
  pluck("data", 1) %>%
  as_tibble()
```

Obtaining PMF values

```
va_md_pmf_data %>%
  glimpse()
```

```
## Observations: 30
## Variables: 17
## $ fill     <chr> "#00BFC4", "#F8766D", "#00BFC4", "#F8766D", "#00BFC4"...
## $ y        <dbl> 0.003731343, 0.003731343, 0.029850746, 0.029850746, 0...
## $ count    <dbl> 1, 0, 8, 0, 7, 0, 7, 0, 26, 5, 11, 2, 16, 1, 11, 4, 1...
## $ x        <dbl> 14, 14, 16, 16, 18, 18, 20, 20, 22, 22, 24, 24, 26, 2...
## $ xmin     <dbl> 13, 13, 15, 15, 17, 17, 19, 19, 21, 21, 23, 23, 25, 2...
## $ xmax     <dbl> 15, 15, 17, 17, 19, 19, 21, 21, 23, 23, 25, 25, 27, 2...
## $ density  <dbl> 0.003731343, 0.000000000, 0.029850746, 0.000000000, 0...
## $ ncount   <dbl> 0.03846154, 0.00000000, 0.30769231, 0.00000000, 0.269...
## $ ndensity <dbl> 10.30769, 0.00000, 82.46154, 0.00000, 72.15385, 0.000...
## $ PANEL    <int> 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,...
## $ group    <int> 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1,...
## $ ymin     <dbl> 0.000000000, 0.003731343, 0.000000000, 0.029850746, 0...
## $ ymax     <dbl> 0.003731343, 0.003731343, 0.029850746, 0.029850746, 0...
## $ colour   <lgl> NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, N...
## $ size     <dbl> 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5...
## $ linetype <dbl> 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,...
## $ alpha    <lgl> NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, N...
```

Obtaining PMF values

To get the Maryland PMF data:

```
md_pmf_data <- va_md_pmf_data %>%
  filter(group == 1) %>%
  select(x, density)
```

x	density
14	0
16	0
18	0
20	0
22	0.104166666666667
24	0.0416666666666667
26	0.0208333333333333
…	…

Obtaining PMF values

To get the Virginia PMF data:

```
va_pmf_data <- va_md_pmf_data %>%
  filter(group == 2) %>%
  select(x, density)
```

x	density
14	0.00373134328358209
16	0.0298507462686567
18	0.0261194029850746
20	0.0261194029850746
22	0.0970149253731343
24	0.041044776119403
26	0.0597014925373134
…	…

Cumulative distribution functions

Data by percentile rank

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- **Advantages**
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  - Different classes of data distributions have distinct shapes
- The **cumulative distribution function** (CDF) lets us map between percentile rank and each value in a data column

Creating CDFs in R

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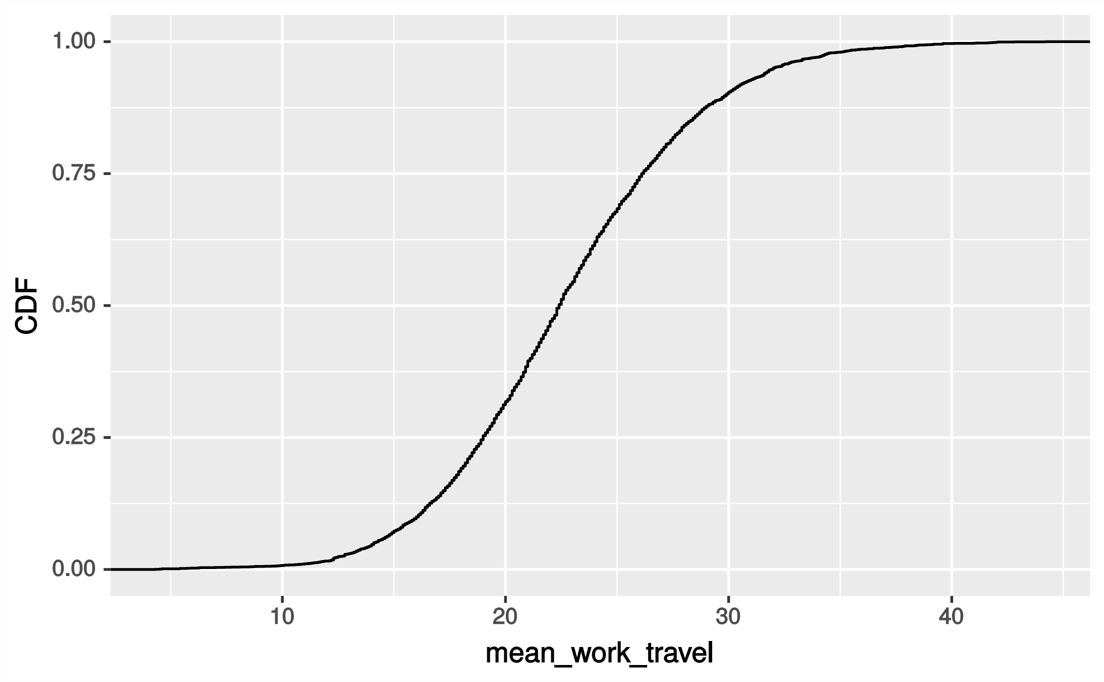
```
county %>%
  ggplot() +
  stat_ecdf(mapping = aes(x = mean_work_travel)) +
  labs(y = "CDF")
```



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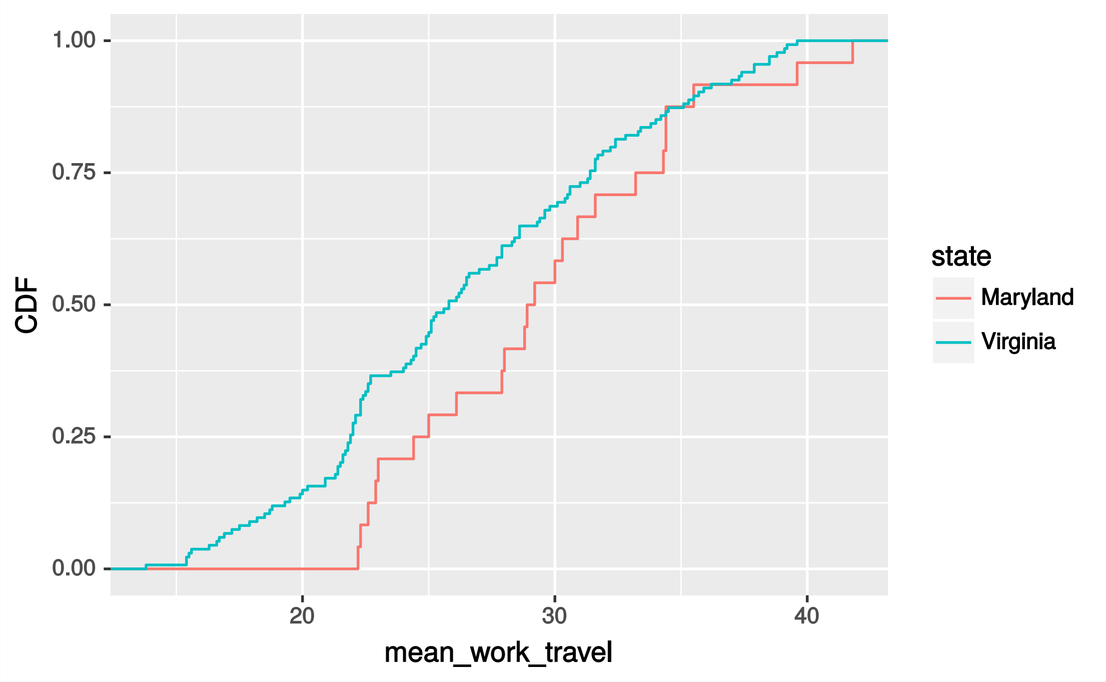


Get CDF data out of plot

Assign the plot to a variable:

```
va_md_cdf_figure <- county %>%
  filter(state == "Virginia" | state == "Maryland") %>%
  ggplot() +
  stat_ecdf(mapping = aes(x = mean_work_travel, color = state)) +
  labs(y = "CDF")
```

Use `ggplot_build()` with `pluck()` and `as_tibble()`:

```
va_md_cdf_df <- va_md_cdf_figure %>%
  ggplot_build() %>%
  pluck("data", 1) %>%
  as_tibble() %>%
  select(group, x, y) %>%
  rename(mean_work_travel = "x", cdf = "y", state = "group") %>%
  mutate(state = recode(state, `1` = "Maryland", `2` = "Virginia")) %>%
  arrange(desc(state), mean_work_travel)
```

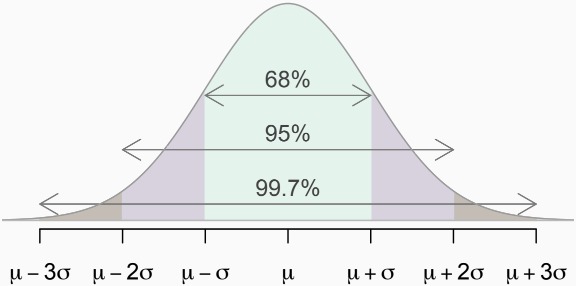
Get CDF data out of plot

state	mean_work_travel	cdf
Virginia	-Inf	0
Virginia	13.8	0.00746268656716418
Virginia	15.4	0.0223880597014925
Virginia	15.5	0.0298507462686567
Virginia	15.6	0.0373134328358209
Virginia	16.3	0.0447761194029851
Virginia	16.6	0.0522388059701493
Virginia	16.7	0.0597014925373134
Virginia	16.9	0.0671641791044776
Virginia	17.2	0.0746268656716418
…	…	…

Analyzing the normal distribution

68-95-99.7 Rule

- For nearly normally distributed data,
  - about 68% falls within 1 SD of the mean,
  - about 95% falls within 2 SD of the mean,
  - about 99.7% falls within 3 SD of the mean.
- It is possible for observations to fall 4, 5, or more standard deviations away from the mean, but these occurrences are very rare if the data are nearly normal.



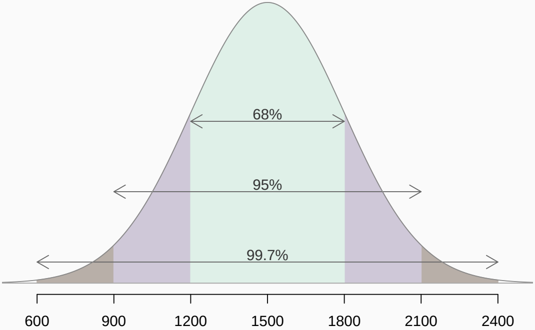
Describing variability using the 68-95-99.7 Rule

SAT scores are distributed nearly normally with mean 1500 and standard deviation 300.

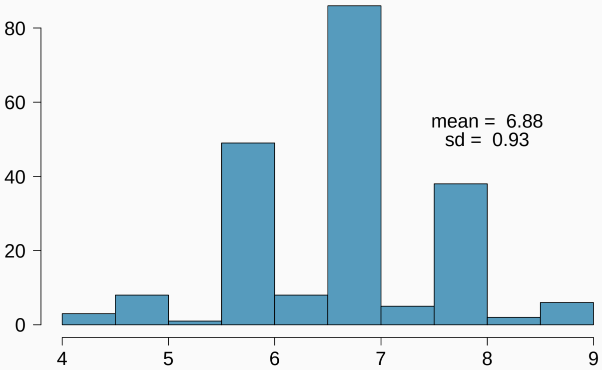
Describing variability using the 68-95-99.7 Rule

SAT scores are distributed nearly normally with mean 1500 and standard deviation 300.

- ~68% of students score between 1200 and 1800 on the SAT.
- ~95% of students score between 900 and 2100 on the SAT.
- ~99.7% of students score between 600 and 2400 on the SAT.



Number of hours of sleep on school nights



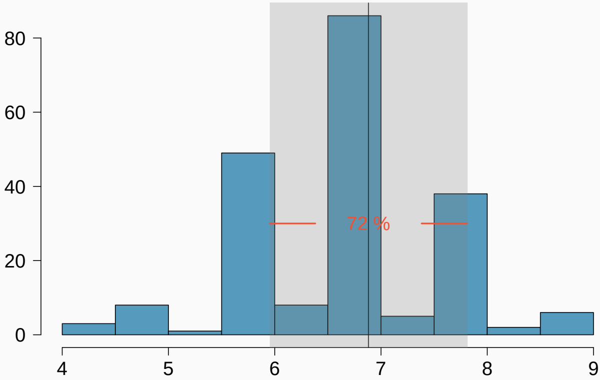
- Mean = 6.88 hours, SD = 0.92 hrs

  72% of the data are within 1 SD of the mean: 6.88 ± 0.93

  92% of the data are within 1 SD of the mean: $6.88 \pm 2 \times 0.93$

  99% of the data are within 1 SD of the mean: $6.88 \pm 3 \times 0.93$

Number of hours of sleep on school nights



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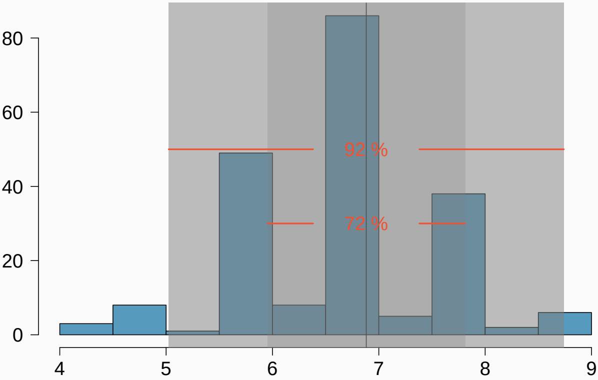

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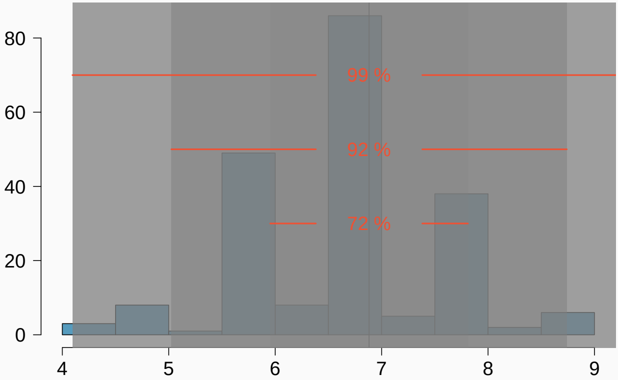

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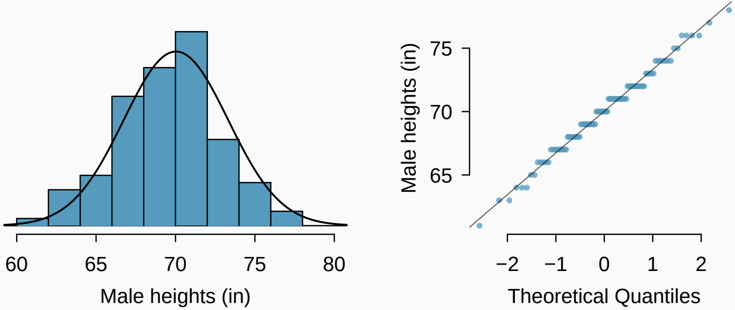
Evaluating the normal approximation

Normal probability plot

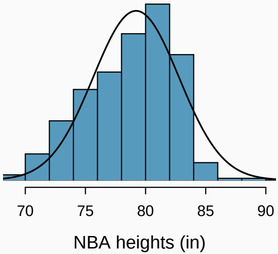
A histogram and *normal probability plot* of a sample of 100 male heights.

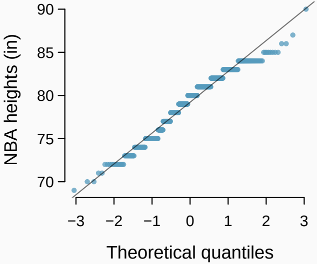


Anatomy of a normal probability plot

- Data are plotted on the y-axis of a normal probability plot, and theoretical quantiles (following a normal distribution) on the x-axis.
- If there is a linear relationship in the plot, then the data follow a nearly normal distribution.
- Constructing a normal probability plot requires calculating percentiles and corresponding z-scores for each observation, which is tedious. Therefore we generally rely on software when making these plots.

Below is a histogram and normal probability plot for the NBA heights from the 2008-2009 season. Do these data appear to follow a normal distribution?





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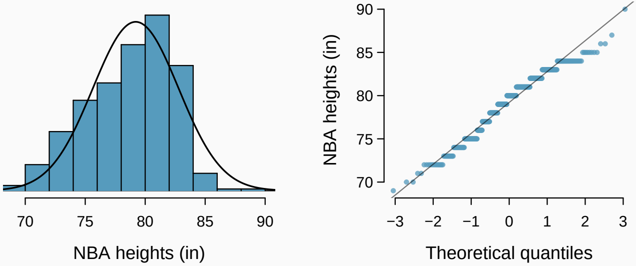
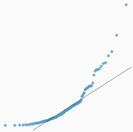
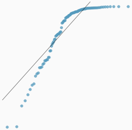


Why do the points on the normal probability have jumps?

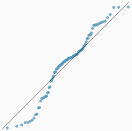
Normal probability plot and skewness



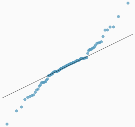
Right skew - Points bend up and to the left of the line.



Left skew- Points bend down and to the right of the line.



Short tails (narrower than the normal distribution) - Points follow an S shaped-curve.



Long tails (wider than the normal distribution) - Points start below the line, bend to follow it, and end above it.

Central limit theorem

Central limit theorem

Review the Central Limit Theorem animation on Seeing Theory

Credits

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